ON A PROBLEM OF THE THEORY OF NONLINEAR OSCILLATIONS

(OB ODNOI ZADACHE TEORII NELINEINYKH KOLEBANII)

PMM Vol.25, No.5, 1961, pp. 963-964

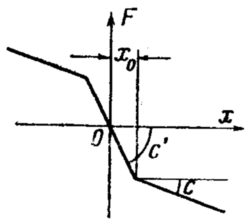
G. N. PUCHENKIN
(Leningrad)

(Received April 18, 1961)

1. The equation for oscillation of a real point under the actions of a sinusoidal force and of a nonlinear restoring force has the form

$$m\frac{d^2x}{dt^2} + cx = \varepsilon f(x) + \varepsilon E \sin \nu t \tag{1.1}$$

Here $f(x)$ is a nonlinear or piecewise linear function of x, and ϵ is a small parameter.



In [1], Equation (1.1) is solved by an asymptotic method, and the first approximation solution is found in the form

$$x = a\cos\psi, \qquad \psi = \nu t + \theta \tag{1.2}$$

The values of a and θ are determined by means of the following system of equations:

$$\frac{da}{dt} = -\frac{\omega_1(a)}{2\pi\omega m} - \frac{\varepsilon E}{m(\omega+\nu)}\cos\theta, \qquad \frac{d\theta}{dt} = \omega - \nu - \frac{\omega_2(a)}{2\pi a m\omega} + \frac{\varepsilon E}{ma(\omega+\nu)}\sin\theta \tag{1.3}$$

where

$$\omega_1(a) = \varepsilon\int_0^{2\pi} f(a\cos\psi)\sin\psi\,d\psi, \qquad \omega_2(a) = \varepsilon\int_0^{2\pi} f(a\cos\psi)\cos\psi\,d\psi, \qquad \omega = \sqrt{\frac{c}{m}} \tag{1.4}$$

If $f(x)$ is a polynomial or a piecewise linear function with a graph that is symmetric with respect to the origin, then $\omega_1(a) = 0$. Equation (1.3) becomes in this case

$$\frac{da}{dt} = -\frac{\varepsilon E}{m(\omega+\nu)}\cos\theta, \qquad \frac{d\theta}{dt} = \omega - \nu - \frac{\omega_2(a)}{2\pi\omega a m} + \frac{\varepsilon E}{ma(\omega+\nu)}\sin\theta \tag{1.5}$$

The system of equations (1.5) is solved in [1] for the stationary synchronous operating conditions with constant amplitude a and with a frequency equal to that of the disturbing force. In this case the derivatives da/dt and $d\theta/dt$ are set equal to zero In this manner one can determine, by means of (1.5), the amplitude of the stationary synchronous oscillations. We shall show that Equations (1.5) make it possible to solve the problem in the general case by finding the solutions $a = a(t)$ and $\theta = \theta(t)$. By Equations (1.5) we have

$$\frac{da}{d\theta}\left[\omega - \nu - \frac{\omega_2(a)}{2\pi\omega a m} + \frac{\varepsilon E}{ma(\omega+\nu)}\sin\theta\right] = -\frac{\varepsilon E}{m(\omega+\nu)}\cos\theta$$

Hence, integrating, we obtain

$$\frac{1}{2}(\omega-\nu)a^2 - \frac{1}{2\pi\omega m}\int\omega_2(a)\,da + \frac{\varepsilon E}{m(\omega+\nu)}a\sin\theta = C \tag{1.6}$$

In this equation, as well as in all following ones, we select only one value of the indefinite integral without the arbitrary constant.

Eliminating θ from the first one of Equation (1.5) by means of (1.6), we find

$$a\left\{\left[\frac{\varepsilon E}{m(\omega+\nu)}\right]^2 a^2 - \left[\frac{1}{2}(\omega-\nu)a^2 - \frac{1}{2\pi\omega m}\int\omega_2(a)\,da - C\right]^2\right\}^{-\frac{1}{2}} da = dt \tag{1.7}$$

After integration of the last equation, we obtain $a = a(t)$, and after that we find by means of (1.6) the function $\theta = \theta(t)$. Equation (1.7) shows that in the general case the amplitude does not tend towards a constant value as $t \to \infty$.

2. Let us consider an example. Suppose that we are given a system with a characteristic restoring force consisting of straight line segments (see figure). For this system

$$\varepsilon f(x) = \begin{cases} -(c'-c)x & \text{for } -x_0 \leqslant x \leqslant x_0 \\ -(c'-c)x_0 & \text{for } x_0 \leqslant x \leqslant \infty \\ (c'-c)x_0 & \text{for } -\infty \leqslant x \leqslant -x_0 \end{cases}$$

Evaluating $\omega_2(a)$ by means of the second formula of (1.4), we divide the interval of integration into three parts. The limits of integration are chosen in each interval in accordance with the first equation (1.2). Making use of a known result [1]

$$\omega_2(a) = -2(c'-c)\left[a\sin^{-1}\frac{x_0}{a} + x_0\sqrt{1-\left(\frac{x_0}{a}\right)^2}\right] \tag{2.1}$$

we obtain

$$\int \omega_2(a)\, da = -2(c'-c)\left[\left(x_0^2 + \frac{1}{2}a^2\right)\sin^{-1}\frac{x_0}{a} + \frac{3}{2}x_0\sqrt{a^2 - x_0^2}\right] \quad (2.2)$$

In particular, if $x_0 = 0$, $c' = \infty$, and $c' x_0 = F_0$, we obtain

$$\int \omega_2(a)\, da = -4F_0 a \quad (2.3)$$

Integrating Equation (1.7) for this particular case, we obtain elliptic integrals in the left part. In accordance with existing solutions [1,2], the resonance ($\omega = \nu$) in the system under consideration leads to an unbounded increase of the amplitude of the oscillations. This, however, is always the case. Indeed, under resonance Equation (1.7), in accordance with Expression (2.2), takes on the form

$$a\left\{\left(\frac{\varepsilon E}{2m\nu}\right)^2 a^2 - \left[\frac{c'-c}{\pi m\nu}\left(\frac{3}{2}x_0\sqrt{a^2 - x_0^2} + \left(x_0^2 + \frac{1}{2}a^2\right)\sin^{-1}\frac{x_0}{a}\right) - C\right]^2\right\}^{-\frac{1}{2}} da = dt \quad (2.4)$$

If the condition

$$\left(\frac{\varepsilon E}{2m\nu}\right)^2 - \left(2\frac{c'-c}{\pi m\nu}x_0\right)^2 > 0, \quad \text{or} \quad \varepsilon E > 4\frac{|c'-c|}{\pi}x_0 \quad (2.5)$$

is satisfied, then Expression (2.4) will be positive when $a \to \pm\infty$, and resonance within the system leads to an unbounded increase of the amplitude of oscillations. If certain inequalities, which are the reverse of (2.5), are satisfied, then the expression in braces of (2.4) will be negative when $a \to \pm\infty$, and the resonance in the system under consideration will not lead to an unbounded increase of the amplitude of the oscillations.

The largest absolute value of the amplitude in this case is found as the largest absolute value of the real roots of the equation

$$\left(\frac{\varepsilon E}{2m\nu}\right)^2 a^2 - \left\{\frac{c'-c}{\pi m\nu}\left[\frac{3}{2}x_0\sqrt{a^2 - x_0^2} + \left(x_0^2 + \frac{1}{2}a^2\right)\sin^{-1}\frac{x_0}{a}\right] - C\right\}^2 = 0 \quad (2.6)$$

For a system with an initial stretching ($x_1 = 0$, $c' = \infty$, $c' z_0 = F_0$), Equation (2.4) can be solved in terms of elementary functions, and the condition of boundedness of the amplitude of oscillations under resonance will be

$$\varepsilon E < 4F_0/\pi \quad (2.7)$$

In this case, Equation (2.6) takes on the form

$$\left(\frac{\varepsilon E}{2m\nu}\right)^2 a^2 - \left(\frac{2F_0}{\pi m\nu}a - C\right)^2 = 0$$

From this we obtain extremal values of the amplitude of oscillations under resonance for a system with initial stretching subjected to the condition (2.7)

$$a_1 = \frac{2Cm\nu}{4F_0/\pi - \varepsilon E}, \qquad a_2 = \frac{2Cm\nu}{4F_0/\pi + \varepsilon E}, \quad C = \frac{2F_0}{\pi m\nu}a + \frac{\varepsilon E}{2m\nu}a\sin\theta \tag{2.8}$$

(Here C is the same as in (1.6) and (2.3); the amplitude of oscillations changes from a_1 to a_2 and back.)

Making use of the second equation in (1.5), of (2.1) and of (2.8), we obtain the cyclical frequency of the oscillations in this case:

$$\frac{d\psi}{dt} = \nu + \frac{d\theta}{dt} = \nu + \frac{C}{a^2}$$

BIBLIOGRAPHY

1. Bogoliubov, N.N. and Mitropol'skii, Iu.A., *Asimptoticheskie metody v teorii nelineinykh kolebanii (Asymptotic Methods in the Theory of Nonlinear Oscillations)*. Fizmatgiz, 1958.

2. Lur'e, A.I. and Chekmarev, A.I., Vynuzhdennye kolebaniia v nelineinoi sisteme s kharakteristikoi iz priamolineinykh otrezkov (Forced oscillations in nonlinear systems with a characteristic consisting of straight line segments). *PMM* Vol. 1, No. 3, 1938.

Translated by H.P.T.